AP* Calculus Review

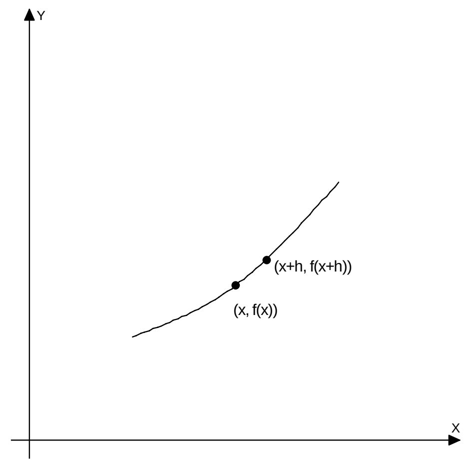
Limits, Continuity, and the Definition of the Derivative

Teacher Packet

DEFINITION **Derivative of a Function**

The **derivative** of the function f with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



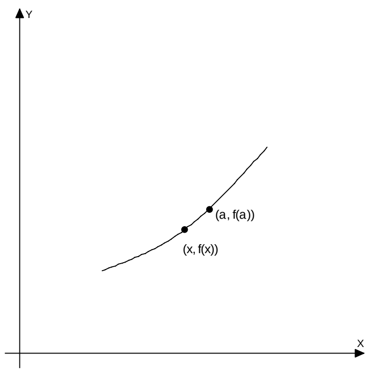
provided the limit exists.

You will want to recognize this formula (a slope) and know that you need to take the derivative of $f(x)$ when you are asked to find $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

DEFINITION (ALTERNATE**)** **Derivative at a Point**

The **derivative** of the function $\boldsymbol{f}$ **at the point** $\boldsymbol{x = a}$ is the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$



provided the limit exists.

This is the slope of a segment connecting two points that are very close together.

DEFINITION **Continuity**

A function f is continuous at a number a if

1) $f(a)$ is defined (a is in the domain of f)

2) $\lim_{x \to a} f(x)$ exists

3) $\lim_{x \to a} f(x) = f(a)$

A function is continuous at an *x* if the function has a value at that *x*, the function has a limit at that *x*, and the value and the limit are the same.

Example:

Given $f(x) = \begin{cases} x^2 + 3, & x \le 2 \\ 3x + 2, & x > 2 \end{cases}$

Is the function continuous at $x = 2$?

$f(x) = 7$

$\lim_{x \to 2^-} f(x) = 7$, but the $\lim_{x \to 2^+} f(x) = 8$

The function does not have a limit as $x \to 2$, therefore the function is not continuous at $x = 2$.

Limits as *x* approaches ∞

For rational functions, examine the *x* with the largest exponent, numerator and denominator. The *x* with the largest exponent will carry the weight of the function.

If the *x* with the largest exponent is in the denominator, the denominator is growing faster as $x \to \infty$. Therefore, the limit is 0.

$$\lim_{x\to\infty} \frac{3+x}{x^4-3x+7} = 0$$

If the *x* with the largest exponent is in the numerator, the numerator is growing faster as $x \to \infty$. The function behaves like the resulting function when you divide the *x* with the largest exponent in the numerator by the *x* with the largest exponent in the denominator.

$$\lim_{x\to\infty} \frac{3+x^5}{x^2-3x+7} = \infty$$

This function has end behavior like $x^3 \left(\dfrac{x^5}{x^2}\right)$. The function does not reach a limit, but to say the limit equals infinity gives a very good picture of the behavior.

If the *x* with the largest exponent is the same, numerator and denominator, the limit is the coefficients of the two *x's* with that largest exponent.

$\lim_{x\to\infty} \dfrac{3+4x^5}{7x^5-3x+7} = \dfrac{4}{7}$. As $x \to \infty$, those x^5 terms are like gymnasiums full of sand. The few grains of sand in the rest of the function do not greatly affect the behavior of the function as $x \to \infty$.

LIMITS

$\lim\limits_{x\to c} f(x) = L$

The limit of f of x as x approaches c equals L.

As x gets closer and closer to some number c (but does not equal c), the value of the function gets closer and closer (and may equal) some value L.

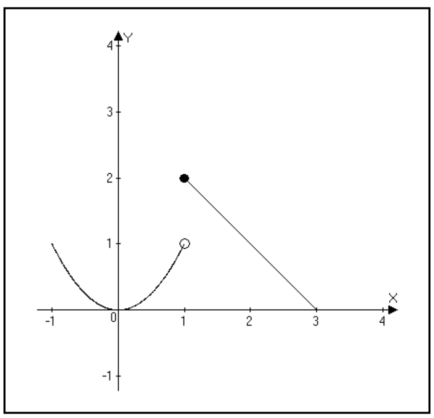
One-sided Limits

$\lim\limits_{x\to c^-} f(x) = L$

The limit of f of x as x approaches c from the left equals L.

$\lim\limits_{x\to c^+} f(x) = L$

The limit of f of x as x approaches c from the right equals L.



Using the graph above, evaluate the following:

$\lim\limits_{x\to 1^-} f(x) =$ $\lim\limits_{x\to 1^+} f(x) =$ $\lim\limits_{x\to 1} f(x) =$

Practice Problems

Limit as *x* approaches infinity

1. $\lim\limits_{x\to\infty}\left(\dfrac{3x-7}{5x^4-8x+12}\right)=$

2. $\lim\limits_{x\to\infty}\left(\dfrac{3x^4-2}{5x^4-2x+1}\right)=$

3. $\lim\limits_{x\to\infty}\left(\dfrac{x^6-2}{10x^4-9x+8}\right)=$

4. $\lim\limits_{x\to\infty}\left(\dfrac{7x^4-2}{5-2x^3-14x^4}\right)=$

5. $\lim\limits_{x\to\infty}\left(\dfrac{\sin x}{e^x}\right)=$

6. $\lim\limits_{x\to-\infty}\left(\dfrac{\sqrt{x^2-9}}{2x-3}\right)=$

7. $\lim\limits_{x\to\infty}\left(\dfrac{\sqrt{x^2-9}}{2x-3}\right)=$

Practice Problems

Limit as *x* approaches a number

8. $\lim_{x \to 2} \left(x^3 - x + 1\right)$

9. $\lim_{x \to 2} \left(\dfrac{x^2 - 4}{x - 2}\right) =$

10. $\lim_{x \to 2^-} \left(\dfrac{3}{x - 2}\right) =$

11. $\lim_{x \to 2^+} \left(\dfrac{3}{x - 2}\right) =$

12. $\lim_{x \to 2} \left(\dfrac{3}{x - 2}\right) =$

13. $\lim_{x \to 2^+} \left(\dfrac{3}{2 - x}\right) =$

14. $\lim_{x \to \frac{\pi}{4}} \left(\dfrac{\sin x}{x}\right) =$

15. $\lim_{x \to \frac{\pi}{4}} \left(\dfrac{\tan x}{x}\right) =$

1. What is $\lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$?

(A) $\sin x$ (B) $\cos x$ (C) $-\sin x$

(D) $-\cos x$ (E) The limit does not exist

2. $\lim_{\Delta x \to 0} \frac{\cos\left(\frac{\pi}{3} + \Delta x\right) - \cos\left(\frac{\pi}{3}\right)}{\Delta x} =$

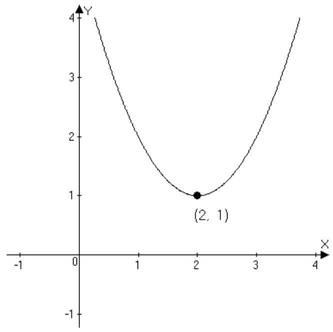
(A) $-\frac{\sqrt{3}}{2}$ (B) $-\frac{1}{2}$ (C) 0

(D) $\frac{1}{2}$ (E) $\frac{\sqrt{3}}{2}$

3. $\lim_{h \to 0} \frac{(x+h)^3 - (x^3)}{h} =$

(A) $-x^3$ (B) $-3x^2$ (C) $3x^2$

(D) x^3 (E) The limit does not exist



4. The graph of $y = f(x)$ is shown above. $\lim\limits_{x \to 2}\left(\left(f(x)^3\right) - 3f(x) + 7\right) =$

(A) 1 (B) 5 (C) 7 (D) 9 (E) Does not exist

5. If $f(x) = \begin{cases} \dfrac{x^2 - 3x - 4}{x+1}, & x \neq -1 \\ 2, & x = -1 \end{cases}$, what is $\lim\limits_{x \to -1} f(x)$?

(A) –5 (B) 0 (C) 2 (D) 3 (E) Does not exist

6. $\lim\limits_{x \to \infty}\left(\dfrac{2x^6 - 5x^3 + 10}{20 - 4x^2 - x^6}\right) =$

(A) –2 (B) $-\dfrac{1}{2}$ (C) $\dfrac{1}{2}$ (D) 2 (E) Does not exist

7. $\lim\limits_{x \to \infty}\left(\dfrac{2x^5 - 5x^3 + 10}{20 - 4x^2 - x^6}\right) =$

(A) –2 (B) $-\dfrac{1}{2}$ (C) 0 (D) $\dfrac{1}{2}$ (E) 2

8. $\lim_{x \to \infty} \left(1 + e^{\frac{1}{2} + \frac{1}{x}} \right) =$

(A) $-\infty$ (B) 0 (C) $e^{\frac{1}{2}}$

(D) $1 + e^{\frac{1}{2}}$ (E) ∞

9. $\lim_{x \to 3^+} \frac{5}{3 - x} =$

(A) $-\infty$ (B) -5 (C) 0

(D) $\frac{5}{3}$ (E) ∞

10. If $\lim_{x \to \infty} \left(\frac{5n^3}{20 - 3n - kn^3} \right) = \frac{1}{2}$, then $k =$

(A) -10 (B) -4 (C) $\frac{1}{4}$ (D) 4 (E) 10

11. Which of the following is/are true about the function g if $g(x) = \frac{(x-2)^2}{x^2 + x - 6}$?

I. g is continuous at $x = 2$
II. The graph of g has a vertical asymptote at $x = -3$
III. The graph of g has a horizontal asymptote at $y = 0$

(A) I only (B) II only (C) III only (D) I and II only (E) II and III only

12. $f(x)=\begin{cases}\sin x, & x<\dfrac{\pi}{4}\\ \cos x, & x>\dfrac{\pi}{4}\\ \tan x, & x=\dfrac{\pi}{4}\end{cases}$

What is $\lim\limits_{x\to\frac{\pi}{4}} f(x)$?

(A) $-\infty$ (B) 0 (C) 1 (D) $\dfrac{\sqrt{2}}{2}$ (E) ∞

13. $\lim\limits_{x\to a}\left(\dfrac{\sqrt{x}-\sqrt{a}}{x-a}\right)=$

(A) $\dfrac{1}{2\sqrt{a}}$ (B) $\dfrac{1}{\sqrt{a}}$ (C) $\sqrt{a}$ (D) $2\sqrt{a}$ (E) Does not exist

14. $\lim\limits_{x\to 0^+}\dfrac{\ln 2x}{2x}=$

(A) $-\infty$ (B) -1 (C) 0 (D) 1 (E) ∞

15. At $x=4$, the function given by $h(x)=\begin{cases}x^2, & x\le 4\\ 4x, & x>4\end{cases}$ is

(A) Undefined
(B) Continuous but not differentiable
(C) Differentiable but not continuous
(D) Neither continuous nor differentiable
(E) Both continuous and differentiable

Free Response 1

Let h be the function defined by the following:

$$h(x)=\begin{cases}|x-1|+3, & 1\le x\le 2\\ ax^2-bx, & x>2\end{cases}$$

a and b are constants.

(a) If $a=-1$ and $b=-4$, is $h(x)$ continuous for all x in $[1,\ \infty]$? Justify your answer.

(b) Describe all values of a and b such that h is a continuous function over the interval $[1,\ \infty]$.

(c) The function h will be continuous and differentiable over the interval $[1,\ \infty]$ for which values of a and b?

Free Response 2 (No calculator)

Given the function $f(x)=\dfrac{x^3+2x^2-3x}{3x^2+3x-6}$.

(a) What are the zeros of $f(x)$?

(b) What are the vertical asymptotes of $f(x)$?

(c) The end behavior model of $f(x)$ is the function $g(x)$. What is $g(x)$?

(d) What is $\lim\limits_{x\to\infty} f(x)$? What is $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)}$?